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Beyond the mean: Estimating consumer demand systems in the tails

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Abstract: The study proposes a novel approach to estimate price demand elasticities at the various levels of expenditures. Through the expectile estimator, the demand system can be estimated not only at the mean, as is generally done when implementing the OLS, but also at the lower and higher levels of expenditure. A simple demand system of equations focusing on five basic goods: Food, Recreation, Clothing, Transport, Rent, using the Canadian Family Expenditure Survey data was estimated. The comparison of the estimated elasticity of each commodity at the selected expectiles is tested to verify the statistical relevance of any difference among the estimates in the tails and those computed at the centre of the conditional distribution. The results show with a strong evidence that the elasticity of Food and Recreation grows across the expectiles. Clothing variations are less evident, while the Transport and Rent elasticity is basically constant across the expectiles.

Keywords: elasticity, expectiles, demand system

Price elasticity of demand (PED) is one of the key concepts in the microeconomic consumer theory. Alfred Marshall (1890) was the first to introduce the PED explicitly, while the first empirical studies date back to the early 1900s (Pigou 1910). Since these pioneering works, a gradually increasing flow of papers on the PED has appeared in the literature, defining and estimating different consumer demand systems.[1] Even if it is impossible to mention all the contributions to this topic, some of them are particularly worthy of mention (Schultz 1938; Leser 1941; Stone 1954; Theil 1967; Barten 1968; Pollak and Wales 1969; Deaton and Muellbauer 1980; Banks et al. 1997; Lewbel and Pendakur 2009). These works generally follow an empirical approach to provide the PED estimates, relegating the economic theory to a normative set of restrictions based on the consumer behaviour axioms (Barten 1964). The great majority of studies specify the demand systems in an expenditure-share format as a linear function of prices and the total expenditure parameters (Barnett and Serletis 2008). Along this route, Deaton and Muellbauer (1980) and their Linear Approximation of the Almost Ideal Demand System (AIDS) provided a remarkable impetus to the literature, since their model is easily estimated by the linear methods.[2]

In these studies, the demand system key parameters on prices and expenditure are estimated in the whole population, while controlling the sample heterogeneity by means of the socio-demographic characteristics or latent groups (Bertail and Caillavet 2008). PED is then straightforwardly calculated at the sample average values or taking the mean values of all calculated elasticities.

However, the demand system key parameters can be estimated not only at the average level of the actual shares/expenditures, which is the general case. They can be considered also in the tails of the conditional distribution that is at the low and/or high values of the shares/expenditures. This approach allows the PED values to be investigated in contexts far from the average, which is the focus of the present analysis.

To the best of our knowledge, few if any studies have dealt with prices having a different effect at

[1]To date, more than 2400 studies have been published on demand systems (Scopus). Indeed, Google Scholar provides an even larger number (50 000).

[2]The more recent EASI (Exact Affine Stone Index) demand system by Lewbel and Pendakur (2009) can be estimated as a linear model as well.

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various levels of consumption. Manning et al. (1995), for example, highlighted a different impact of the wine price on light, moderate, or heavy drinkers, estimating a single expenditure function by quantile regression. However, to date no studies have provided the PED values at different levels of consumption – i.e. different quantiles or expectiles – computed from a formal and theoretically consistent system of demand equations.

Estimating the PED at different levels cannot be considered a mere empirical exercise since it is undeniable that a systematic low or high level of consumption might mirror different consumption habits directly or indirectly. For instance, the well-being deprivation can be related to a high share of essential goods and a low share of luxury goods, while the economic opulence can be linked to a low share of essential goods and high share of luxury goods. Thus going beyond the mean allows the PED estimates to account for different kinds of consumption. Even though the PED is expected to be inelastic for the essential goods and more elastic for the luxury goods, it seems plausible to expect that both elasticities become more inelastic at a higher budget share, reflecting the consumer response to price in a completely different situation. In other words, the different responses to price, related to the sample heteroskedasticity, entails the estimation of the PED at the various levels of consumption.

In applied economics, the analysis in the tails is frequently implemented to measure earnings and wage differentials (Buchinsky 1994; Katz and Autor 1999; Angrist et al. 2006), or the impact of policy changes. These models are generally estimated via quantile regressions (Koenker and Bassett 1978; Koenker 2005) or via expectiles (Newey and Powell 1987; Sobotka et al. 2013). Both quantile and expectile regressions allow the estimation of parameters away from the conditional mean, thus revealing the impact of the explanatory variables at lower (higher) values than the conditional mean of the dependent variable.

The idea is that the estimated model may have differing coefficients depending upon the selected point of the conditional distribution: the centre, the upper or the lower tail. At the lower or upper tail of the data generating process, the link between the dependent and explanatory variables, shares versus prices, may diverge from the relationship estimated at the centre of the distribution, the conditional mean. To move away from the conditional mean, quantile and expectile estimators modify the objective function of the ordinary least squares (OLS) estimator. The quantile regression approach defines the objective function as the weighted sum of the absolute value of the errors, while the expectile estimator considers the weighted sum of squared errors. Both estimators introduce an asymmetric weighting system that moves the estimated regression along the conditional distribution, toward the tails and away from the centre.

The absolute value function in the quantile regression estimator grants robustness with respect to outliers in the dependent variable. On the other hand, dealing with the cross-equation constraints is troublesome in the quantile regression setting (Davino et al. 2013). The expectile regression, which considers the asymmetrically weighted sum of squared errors, is easier to compute than the quantile regressions, and the cross-equation restrictions are easily dealt with[3]. The presence of restrictions based on the consumer behaviour axioms leads to selecting the expectile estimator.

In the following sections, the empirical analysis considers a simple demand system of equations focusing on five basic goods: Food, Recreation, Clothing, Transport, Rent, using the Canadian Family Expenditure Survey data in Lewbel and Pendakur (2009). The estimated model has a smaller number of equations than the one presented by Lewbel and Pendakur (2009). Indeed, the purpose is not to estimate of the model *per se*, but to analyse its behaviour at the various expectiles. This allows us to check whether the estimated coefficients do actually change in the tails with respect to the simple OLS (conditional mean) results.

Once the model has been computed at various expectiles, the comparison of the estimated elasticity of each commodity at the selected expectiles is tested to verify the statistical relevance of any difference among the estimates in the tails and those computed at the centre of the conditional distribution. To this end, a Wald type test is implemented to verify the null of equality of the elasticity estimates across expectiles.

The results show with a strong evidence that the elasticity of Food and Recreation grows across the expectiles. Clothing variations are less evident, while the Transport and Rent elasticity is basically constant across the expectiles.

[3]The lack of robustness typical of the OLS can be solved by a preliminary analysis of anomalous values, estimating the model in a truncated sample, where the detected outliers have been removed.

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THEORETICAL FRAMEWORK: EXPECTILE AIDS

A vast theoretical and empirical literature has devoted a huge effort to estimating the demand system and the resulting PED (Barnett and Serletis 2008). The most widely used models adopt a specification of the system of demand equations in which the dependent variables are the budget shares of the consumption categories of interest. Among them, the linear approximation to the Almost Ideal Demand System (LA-AIDS) of Deaton and Muellbauer (1980) has enjoyed immense popularity because it can be estimated using linear models (Dossche et al. 2010).

The present analysis focuses on the LA-AIDS, although several caveats are known in the literature, such as the assumption of the linear Engel curves. As mentioned above, since the aim of the paper is to analyse the behaviour of the model in the tails of the distribution, the adoption of non-linear models such as the QUAIDS (Banks et al. 1997; Christensen 2014) would somehow cloud the picture. A future research will be devoted to the analysis of non-linear demand equations.

For each household h, the equation of the budget share w_i of the i^{th} good in the LA-AIDS model may be written as:

$$w_{ih} = \alpha_{ih*} + \sum_{j=1}^{n} \gamma_{ij} \ln P_{jh} + \beta_i \ln\left(\frac{X_h}{P_h^*}\right) + u_{ih}$$
$$i, j = 1,\dots, n;\ h = 1,\dots H \qquad (1)$$

where P_{ih} is the price of the i^{th} good for the h^{th} household, $X_h = \sum_{i=1}^{n} P_{ih} q_{ih}$ is the total expenditure for the household, q_{ih} is the consumed quantity with the share computed as $w_{ih} = P_{ih}q_{ih}/X_h$. The terms α_{ih*}, β_i and γ_{ij} are the parameters to be estimated, respectively the intercept, the total expenditure coefficient, the own price coefficients when $i = j$, and the cross price coefficients for $i \neq j$, while u_{ih} is the error term. Following many other empirical studies (Mizobuchi and Tanizaki 2013) a corrected Stone Index[4] could be selected as the deflating price index, P_h^*, using the average share $\overline{w}_i$:

$$\ln P_h^* = \sum_{i=1}^{n} \overline{w}_i \ln P_{ih} \quad h = 1, \dots, H \qquad (2)$$

To account for the possible heterogeneity in household preferences and expenditure behaviour over time, the intercept is expressed as a linear function of $k = 1, \dots, K$ socio-demographic and time variables.

$$\alpha_{ih*} = \alpha_i + \sum_{k=1}^{K} \delta_{ik} D_{kh} \qquad (3)$$

Finally, the usual parameter restrictions on symmetry (4a), homogeneity (4b) and additivity (4c) are imposed in estimating the demand system parameters, as specified below:

$$\gamma_{ij} = \gamma_{ji} \quad \forall\, i,j \qquad (4a)$$

$$\sum_j \gamma_{ij} = 0 \quad \forall\, i,j \qquad (4b)$$

$$\sum_i \beta_i = 0;\ \sum_i \alpha_i = 1;\ \sum_i \gamma_{ij} = 0 \ \ \forall j\ ;\sum_i \delta_{ik} = 0 \ \ \forall k \qquad (4c)$$

In the expectile setting, equation (1) is modified to include an asymmetric weighting system that moves the stochastic equation along the conditional distribution, away from the conditional mean and up or down toward the tails. Equation (1) becomes:

$$g_{ih} w_{ih} = g_{ih}\left[\alpha_{ih*} + \sum_{j=1}^{n} \gamma_{ij} \ln P_{jh} + \beta_i \ln\left(\frac{X_h}{P_h^*}\right)\right] + g_{ih} u_{ih}$$
$$i, j = 1,\dots, n;\ h = 1,\dots, H \qquad (5)$$

where the asymmetric weighting system is

$$g_{ih} = \begin{cases} \theta & \text{if } \ u_{ih} > 0 \\ 1-\theta & \text{otherwise} \end{cases}$$

For instance, to compute the θ = 25th expectile g_{ih} assigns weights g_{ih} = 0.75 to those observations below the regression, in order to attract toward the lower tail, the estimated equation, and assigns weights g_{ih} = 0.25 to the observations in the upper tail of the conditional distribution.

At a given expectile (suppressing the household subscript h), the definition of PED is:

$$\eta_{i,j}(\theta) = -\tau_{ij} + \frac{\gamma_{ij}(\theta)}{\overline{w}_i} - \frac{\beta_i(\theta)}{\overline{w}_i}\frac{d\ln P^*}{d\ln P_j} \qquad (6)$$

where τ_{ij} is the Kronecker delta (τ_{ij} = 1 for $i = j$; τ_{ij} = 0 for $i \neq j$) and $\overline{w}_i$ is the sample average[5].

By assuming (Alston et al. 1994), $\frac{d\ln P^*}{d\ln P_j} = w_j$, PED becomes:

$$\eta_{i,j}(\theta) = -\tau_{ij} + \frac{\gamma_{ij}(\theta)}{\overline{w}_i} - \frac{\beta_i(\theta)}{\overline{w}_i} w_j \qquad (7)$$

[4]As shown in Moschini (1995), the usual Stone index is invariant to changes in units of measurements and this may lead to biased parameter estimates. Using the average share $\overline{w}_i$ instead of household share w_{hi} solves this issue.

[5]By using $\overline{w}_i$, $\eta_{i,j}(\theta)$ are estimated at sample mean, which is equivalent to computing the expected value of $\eta_{ij}(\theta)$.

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DATA

The system of demand equations and PED are estimated for the Canadian Family Expenditure Survey data in Lewbel and Pendakur (2009). The original sample comprised nine commodities: food-in, food-out, rent, clothing, household operation, household furnishing and equipment, transportation operation, recreation, and personal care. The sample includes 4847 observations on the quantities and prices collected in the period 1969–1996. The data set also includes a series of five demographic characteristics: (1) age minus 40; (2) a gender dummy assuming value one for men; (3) a car-nonowner dummy equal to one if gasoline expenditures (at 1986 gasoline prices) are less than \$50; (4) a social assistance dummy equal to one if government transfers are greater than 10 percent of gross income; and (5) a time variable equal to the calendar year minus 1986 (equal to zero in 1986). These demographic variables define the intercept α_{ih^*} of equation 3 and are gathered in D_k, for $k = 1, ..., 5$. A detailed description of the data set can be found in Lewbel and Pendakur (2009: 839–840)[6].

This study primarily focuses on the behaviour of the PED in the tails of the conditional distribution. To this end, the demand equation system is simplified by selecting the following five categories with the highest expenditure share: food-in, rent, clothing, transportation and recreation. This choice was entirely driven by the need to reduce the computational burden and to economize on the number of model parameters. The last two columns of Table 1 report the sample mean and standard deviation for the variables of interest.

ESTIMATION

Consider the demand system specified in equation (1) as a generic demand system $\mathbf{u}_h = \mathbf{r}(\mathbf{w}_h, \mathbf{X}_h, \beta) = \mathbf{w}_h - \mathrm{f}(\mathbf{X}_h, \beta)$, not necessarily linear, where $\mathbf{w}_h$ and $\mathbf{u}_h$ are n × 1 vectors, $\mathbf{X}_h$ is a n × L matrix – where L is the number of covariates, and β is a M × n matrix – where M is the number of unknown parameters. We assume that the error vector $\mathbf{u}_h$ is independent and identically distributed over h, with the covariance matrix Ω, where the components of $\mathbf{u}_h$ for given h may be correlated with the variances and covariances that vary over h. The β estimator is defined as:

$$\hat{\beta} \equiv \mathrm{argmin}_{\beta} \left[\mathbf{w} - \mathbf{f}(\mathbf{X},\beta)\right]' \mathbf{\Omega}^{-1} \left[\mathbf{w} - \mathbf{f}(\mathbf{X},\beta)\right] \quad (8)$$

Table 1. Summary statistics for the variables in the demand system

Variable		Mean (H_{tr})	S.D. (H_{tr})	Mean (H)	S.D. (H)
w_1	Food-in expenditure share	0.183	0.092	0.183	0.096
w_2	Renting expenditure share	0.101	0.083	0.103	0.089
w_3	Clothing expenditure share	0.105	0.072	0.106	0.076
w_4	Transportation expenditure share	0.145	0.095	0.146	0.098
w_5	Recreation expenditure share	0.467	0.130	0.462	0.134
p_1	Price of Food-in	1.027	0.314	1.026	0.315
p_2	Price of Renting	0.991	0.324	0.989	0.325
p_3	Price of Clothing	1.096	0.307	1.095	0.308
p_4	Price of Transportation	1.040	0.438	1.037	0.439
p_5	Price of Recreation	1.079	0.341	1.077	0.342
X	Total expenditure	0.919	0.644	0.919	0.650
z_1	Age minus 40	0.701	11.873	0.711	11.888
z_2	Sex (1 if male)	0.510	0.500	0.506	0.500
z_3	Car non-owner (1 if non-owner)	0.413	0.492	0.415	0.493
z_4	Social assistance (1 if presence of social assistance)	0.268	0.443	0.268	0.443
z_5	Time variable (year minus 1986)	3.036	8.703	2.991	8.729

Original sample of size $H = 4847$; truncated sample of size $H_{tr} = 4739$

[6]A potential source of endogeneity may arise from the demographic variables. For instance, complex links may occur between consumption and social assistance or car-nonownership and transportation expenditure. However, as discussed in Lewbel and Pendakur (2009, p. 837) the estimates accounting for endogeneity do not substantially differ from the results computed without instruments. Future studies may generalize the expectile approach to include instrumental variables.

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The solution for $\boldsymbol{\beta}$ and $\boldsymbol{\Omega}$ is computed by the iterative Aitken estimator (Barnett 1976; Kmenta 1986), iterating β and $\boldsymbol{\Omega}$ until convergence, where the latter is computed as $\hat{\boldsymbol{\Omega}} = \frac{1}{H}\sum_{h=1}^{H}\hat{\mathbf{u}}_h\hat{\mathbf{u}}_h'$ and is initialized by $\Omega = \mathbf{I}$.

Having estimated the model at the centre of the conditional distribution, the goal is to move away from it toward the tails. Consider as a first approximation a single equation approach. As mentioned above, the expectiles, or the asymmetric least squares, modify the standard OLS objective function by introducing an asymmetric weighting system that moves the estimated regression along the conditional distribution, away from the conditional mean and up or down toward the tails of the distribution. The objective function of the expectile regression is given by

$$\sum_{u_{ih}>0} \theta\, u_{ih}^2 + \sum_{u_{ih}<0}(1-\theta)\, u_{ih}^2 = \sum_H g_{ih}\, u_{ih}^2 \qquad (9)$$

where g_{ih} is equal to θ or 1– θ according to the position of the error term, above or below the regression.

Going back to the system of demand equations, this involves defining the objective function (8) in terms of $\mathbf{u}_h^* = \mathbf{g}_h u_h = \mathbf{g}_h \mathbf{w}_h - f(\mathbf{g}_h \mathbf{X}_h, \beta(\theta)) = \mathbf{w}_h^* - f(\mathbf{X}_h^*, \beta(\theta))$, yielding

$$\hat{\beta}(\theta) \equiv \operatorname{argmin}_{\beta(\theta)} \left[\mathbf{w}^* - \mathbf{f}\left(\mathbf{X}^*, \beta(\theta)\right)\right]' \Omega^{*-1} \left[\mathbf{w}^* - \mathbf{f}\left(\mathbf{X}^*, \beta(\theta)\right)\right] \qquad (10)$$

with $\Omega^* = \frac{1}{H}\sum_{h=1}^{H} u_h^*\, u_h^{*\prime}$.

Next, the model can be estimated at different values of θ. Our analysis considers the 25th, 35th, 50th, 65th, and 75th expectiles[7]. These estimates are compared with one another, particularly those computed at the 25th, 50th and 75th expectile, and are tested to verify the statistical relevance of their difference. In detail, to test whether the changes in elasticity across expectiles are statistically relevant, the 25th versus the 50th, the 50th versus the 75th, and the 25th versus the 75th expectile are compared. Analogously to the changing coefficient test discussed in Koenker and Basset (1982), the test function is a Wald type χ^2 test. The null hypothesis is H_0: $\eta(\theta_v) = \eta(\theta_m)$, where $\eta(\theta_v)$ and $\eta(\theta_m)$ are the vectors of elasticity estimated at the $\theta = v^{th}$ and at the $\theta = m^{th}$ expectile. The test function is given by

$$\boldsymbol{W} = [\eta(\theta_v) - \boldsymbol{\eta}(\boldsymbol{\theta}_m)]'\, \boldsymbol{\Sigma}^{-1}[\boldsymbol{\eta}(\boldsymbol{\theta}_v) - \boldsymbol{\eta}(\boldsymbol{\theta}_m)] \qquad (11)$$

and is asymptotically distributed as a χ^2 with degrees of freedom equal to the number of comparisons under test.

To implement the $\boldsymbol{W}$ test, the covariances between each pair of expectile regressions are needed. The diagonal elements of the variance covariance matrix across expectiles, Σ, are defined as

$$\sigma_{vv}^2 = E(\eta_{i,i}(\theta_v) - E\eta_{i,i}(\theta_v))^2 \qquad (12)$$

while the off diagonal terms are given by

$$\sigma_{vm}^2 = E[\eta_{i,i}(\theta_v) - E\eta_{i,i}(\theta_v)]\,[\eta_{i,i}(\theta_m) - E\eta_{i,i}(\theta_m)] \qquad (13)$$

The $\boldsymbol{\Sigma}$ matrix is unknown, but it can be estimated by bootstrap. Data on prices, quantities and expenditure are re-sampled T = 1000 times and the 25th, 50th and 75th estimates are computed each time. The estimated coefficients allow the series of T = 1000 estimated PED to be computed for each good and at all the selected expectiles. The variance covariance matrix across expectiles $\boldsymbol{\Sigma}$ can then be computed.

EMPIRICAL RESULTS

In this paper, the expectile regression estimator is implemented to compute the model at the centre and in the tails of the conditional distribution. As stated above, the expectiles are defined as the asymmetrically weighted least squares, and are thus affected by outliers just like the OLS. Therefore, due to the lack of robustness of the expectiles, a preliminary analysis of the OLS residual – the residuals of the regression estimated at the conditional mean – is implemented to detect outliers and to evaluate their impact on the estimated coefficients. Anomalous values can be very damaging in a regression model, since they may be leverage points and as such they cause biased estimates. Once the outliers are detected, they can be discarded and the model can be re-estimated in the truncated data set. By eliminating the outliers, the expectile regression estimator gains robustness, while without this preliminary analysis the results are influenced by the anomalous values and may be biased.

The initial step is to estimate the model as in equation (8). Next, the analysis focuses on the detection of

[7]Just as in the conditional mean analysis (OLS), the demand system is estimated for one expectile at the time. The demand restrictions hold within each expectile and they are not imposed simultaneously across all the expectiles. The results refer to households being in same θ-*th* expectile in the consumption of each *i-th* good.

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anomalous values.[8] The absolute value of the residuals are standardized and are compared to a chosen bound. The values exceeding the bound signal the outlying observations, which are then discarded from the sample. The bound is set by looking at the extreme tails of the standard normal, with p-value $\alpha = 0.0002$, which corresponds to $z = 3.7$.[9] This criterion leads to drop 108 observations from the sample, yielding a truncated sample of size $H_{tr} = 4739$, which amounts to a trimming rate of about 2.2%. The first two columns in Table 1 report the sample mean and standard deviation in the truncated sample H_{tr}.

The above detection rule was double-checked by implementing a single equation outlier detection approach. Each equation is robustly estimated at the conditional mean by the robust M-estimator, defined as $\Sigma_{h=1,..H}\rho(u_{hi})$ where the $\rho(u_{hi})$ function bounds the anomalous values of the sample. The estimator is implemented iteratively, updating at each step the bounding function and the residuals.[10] The final bounding function provides a valuable detection tool to spot the large and influential outliers. In the single equation analysis, the M-estimator points out 127 outlying observations which it is advisable to exclude from the sample. The outliers detected by this single equation approach mostly coincide with the data points previously discarded by implementing the standardized residuals approach based on the initial estimates of the demand equations system in (8). With respect to the standardized residuals, the robust single equation detection rule discards an additional number of 19 observations. However, the system approach to the outlier detection was preferred and these 19 outliers are included in the sample.

Next, the model is re-estimated in the truncated sample and the results are reported in Table 2.[11] The final results on PED in the H_{tr} sample are in the third column of Table 5.

The model can then be estimated at other points of the conditional distribution besides the conditional mean by implementing the expectile regression estima-

Table 2. Demand system estimates at the 50th expectile, H_{tr} sample

	Food-in	Recreation	Clothing	Transport	Rent
α_i	0.416	–0.121	–0.076	0.225	0.556
	33.64	*–10.47*	*–7.4*	*20.17*	*37.65*
β_i	–0.064	0.061	0.045	–0.004	–0.037
	–20.75	*21.2*	*17.84*	*–1.62*	*–9.9*
$\gamma_{i,j: \text{Food-in}}$	0.082	–0.037	–0.056	–0.021	0.032
	6.73	*–3.77*	*–5.56*	*–2.39*	*3.38*
$\gamma_{i,j: \text{Recreation}}$	–0.037	–0.003	0.084	–0.015	–0.029
	–3.77	*–0.17*	*6.36*	*–1.51*	*–2.92*
$\gamma_{i,j: \text{Clothing}}$	–0.056	0.084	0.012	0.013	–0.054
	–5.56	*6.36*	*0.75*	*1.37*	*–5.73*
$\gamma_{i,j: \text{Transport}}$	–0.021	–0.015	0.013	0.023	0.000
	–2.39	*–1.51*	*1.37*	*1.8*	*0.02*
$\gamma_{i,j: \text{Rent}}$	0.032	–0.029	–0.054	0.000	0.050
	3.38	*–2.92*	*–5.73*	*0.02*	*3.56*
$\delta_{i,k: \text{Age}}$	0.001	–0.012	0.026	0.016	0.001
	14.61	*–5.52*	*10.35*	*5.44*	*6.17*
$\delta_{i,k: \text{Sex}}$	–0.001	–0.029	0.016	–0.006	–0.001
	–12.27	*–14.49*	*7.04*	*–2.21*	*–6*
$\delta_{i,k: \text{Car}}$	–0.001	0.035	0.016	–0.017	–0.004
	–8.83	*20.15*	*7.98*	*–7.61*	*–18.33*
$\delta_{i,k: \text{Soc. ass.}}$	0.000	–0.011	–0.122	–0.013	0.000
	–5.42	*–5.88*	*–54.89*	*–5.03*	*–1.81*
$\delta_{i,k: \text{Time}}$	0.001	0.018	0.064	0.020	0.004
	7.08	*6.76*	*21.49*	*6.06*	*15.22*

t-statistics in italics

[8]Table A.1 in the Appendix reports the estimates of the initial model.

[9]The 3.7 bound is quite large, it coincides with the interval ±3.7σ, where σ = 1 in the standard normal, and is wider than the usual ±2σ or ±3σ bounds. This means that only the very large (small) values are discarded.

[10]The Huber (1964) function is defined as $\rho(u_{ih}) = 0.5u_{ih}^2$ for $|u_{ih}| \leq c$ and $\rho(u_{ih}) = c|u_{ih}| - 0.5c^2$ for $|u_{ih}| > c$. It down weights those values exceeding the selected bound c, which is usually set equal to $c = 1.345$. After a first set of iteration, a different bounding function is implemented, the redescending function (Beaton and Tukey 1974), which completely excludes the extreme outlying observations from the data set. This function is defined as $\rho(e_{ih}) = 1/6[1 - (1 - e_{ih}^2)^3]$ for $|e_{ih}| \leq 1$ and $\rho(e_{ih}) = 1/6$ for $|e_{ih}| > 1$, where e_{ih} are the residuals standardized by $e_{ih} = u_{ih}/bS$, with b ranging from 6 to 12 and S being the median absolute deviation of u_{ih}. By looking at the combination of the Huber and the redescending M-estimators, the extreme influential outliers in the data set may be detected and excluded.

[11]Table A.1 in the Appendix provides the results computed in the original sample so that it is possible to compare the estimated coefficients computed in the full sample of size $H = 4847$ with the results of the truncated one, $H_{tr} = 4739$, here discussed. When compared with the results of Table A.1, the estimates and the t statistics of Table 2 are slightly smaller. As a consequence, the null is no longer rejected for some of the estimated coefficients.

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Table 3. Demand system estimates at the 25th expectile, H_{tr} sample

	Food-in	Recreation	Clothing	Transport	Rent
α_i	0.360	–0.049	–0.018	0.191	0.517
	17.49	*–2.83*	*–1.08*	*10.12*	*26.50*
β_i	–0.046	0.040	0.029	–0.001	–0.022
	–9.01	*9.29*	*6.99*	*–0.23*	*–4.48*
$\gamma_{i,j: \text{Food-in}}$	0.061	–0.023	–0.041	–0.025	0.028
	2.95	*–1.54*	*–2.50*	*–1.72*	*2.00*
$\gamma_{i,j: \text{Recreation}}$	–0.023	–0.011	0.066	–0.012	–0.020
	–1.54	*–0.50*	*3.39*	*–0.78*	*–1.44*
$\gamma_{i,j: \text{Clothing}}$	–0.041	0.066	0.004	0.013	–0.042
	–2.50	*3.39*	*0.15*	*0.88*	*–2.97*
$\gamma_{i,j: \text{Transport}}$	–0.025	–0.012	0.013	0.030	–0.006
	–1.72	*–0.78*	*0.88*	*1.45*	*–0.36*
$\gamma_{i,j: \text{Rent}}$	0.028	–0.020	–0.042	–0.006	0.040
	2.00	*–1.44*	*–2.97*	*–0.36*	*2.12*
$\delta_{i,k: \text{Age}}$	0.001	–0.011	0.012	0.007	0.001
	5.40	*–3.10*	*2.95*	*1.52*	*2.17*
$\delta_{i,k: \text{Sex}}$	–0.001	–0.022	0.004	–0.004	–0.001
	–5.91	*–7.61*	*1.35*	*–1.00*	*–2.53*
$\delta_{i,k: \text{Car}}$	0.000	0.024	0.009	–0.011	–0.003
	–4.07	*8.53*	*2.86*	*–3.04*	*–8.80*
$\delta_{i,k: \text{Soc. ass.}}$	0.000	–0.005	–0.092	–0.008	–0.001
	–2.58	*–1.62*	*–24.15*	*–1.94*	*–1.43*
$\delta_{i,k: \text{Time}}$	0.001	0.015	0.067	0.016	0.004
	5.15	*4.44*	*17.40*	*3.70*	*9.22*

t-statistics in italics

Table 4. Demand system estimates at the 75th expectile, H_{tr} sample

	Food-in	Recreation	Clothing	Transport	Rent
α_i	0.377	–0.094	–0.051	0.229	0.539
	36.76	*–10.86*	*–6.71*	*22.91*	*29.95*
β_i	–0.054	0.058	0.040	–0.008	–0.036
	–21.13	*27.18*	*21.62*	*–3.02*	*–7.87*
$\gamma_{i,j: \text{Food-in}}$	0.080	–0.041	–0.051	–0.005	0.017
	8.57	*–5.65*	*–7.01*	*–0.67*	*1.91*
$\gamma_{i,j: \text{Recreation}}$	–0.041	0.021	0.063	–0.016	–0.027
	–5.65	*1.89*	*6.46*	*–2.12*	*–3.07*
$\gamma_{i,j: \text{Clothing}}$	–0.051	0.063	0.024	0.004	–0.040
	–7.01	*6.46*	*2.10*	*0.62*	*–5.34*
$\gamma_{i,j: \text{Transport}}$	–0.005	–0.016	0.004	0.024	–0.007
	–0.67	–2.12	0.62	2.05	–0.64
$\gamma_{i,j: \text{Rent}}$	0.017	–0.027	–0.040	–0.007	0.057
	1.91	*–3.07*	*–5.34*	*–0.64*	*3.20*
$\delta_{i,k: \text{Age}}$	0.001	–0.012	0.024	0.015	0.001
	16.18	*–6.26*	*11.40*	*6.30*	*4.57*
$\delta_{i,k: \text{Sex}}$	–0.001	–0.027	0.015	0.001	–0.002
	–10.60	*–16.64*	*8.23*	*0.31*	*–9.80*
$\delta_{i,k: \text{Car}}$	0.000	0.029	0.011	–0.014	–0.003
	–7.38	*20.78*	*6.86*	*–7.90*	*–19.34*
$\delta_{i,k: \text{Soc. ass.}}$	0.000	–0.013	–0.101	–0.009	0.000
	–5.15	*–7.13*	*–50.81*	*–4.07*	*–1.84*
$\delta_{i,k: \text{Time}}$	0.000	0.022	0.051	0.008	0.005
	1.75	*6.58*	*13.39*	*1.78*	*13.45*

t-statistics in italics

Table 5. Estimated elasticity at the selected expectiles, sample size H_{tr}

Sample H_{tr}	25th	35th	50 th	65 th	75 th
Food-in	–0.586***	–0.501***	–0.442***	–0.421***	–0.467***
	(0.103)	*(0.055)*	*(0.076)*	*(0.102)*	*(0.060)*
Recreation	–1.183	–1.127	–1.095	–0.851	–0.785*
	(0.205)	*(0.164)*	*(0.213)*	*(0.220)*	*(0.169)*
Clothing	–0.988	–0.876	–0.911	–0.792	–0.773*
	(0.208)	*(0.147)*	*(0.201)*	*(0.219)*	*(0.157)*
Transport	–0.789***	–0.776***	–0.834**	–0.891*	–0.823**
	(0.118)	*(0.076)*	*(0.097)*	*(0.123)*	*(0.086)*
Rent	–0.890***	–0.886***	–0.853***	–0.824***	–0.842***
	(0.038)	*(0.031)*	*(0.032)*	*(0.044)*	*(0.039)*

Bootstrapped standard deviation in parenthesis; the stars signal the significance level for the rejection of the null H_0: $|\boldsymbol{\eta}(\theta)| = 1$, *** for $\alpha = 0.01$,** for $\alpha = 0.05$, * for $\alpha = 0.10$

tors[12]. In what follows, the equations are computed at the 25th, 35th, 50th, 65th and 75th expectile. In Table 3 are the estimates for the 25th expectile and Table 4 displays the results for the 75th expectile.[13] Table 5 reports the estimated PED at the various expectiles together with the significance level in rejecting the null H_0: $|\eta(\theta)| = 1$. The table shows that the PED does indeed change across the expectiles. In particular, there is an increasing pattern for Food, Recreation and Clothing, where food rejects the null at each expectile while Recreation and Clothing reject the null only at the highest expectile. In other terms, the Food PED is inelastic at all the considered expectiles. The PED becomes inelastic at the 75th expectile in the case of Recreation and Clothing, while the PED presents a basically stable elastic pattern for Rent and Transportation, rejecting the null hypothesis at every expectile.

[12]Once outliers have been dropped from the sample, the expectile estimator gains robustness and it is no longer affected by anomalous values.

[13]The results for the 35th and the 65th expectile are available on request.

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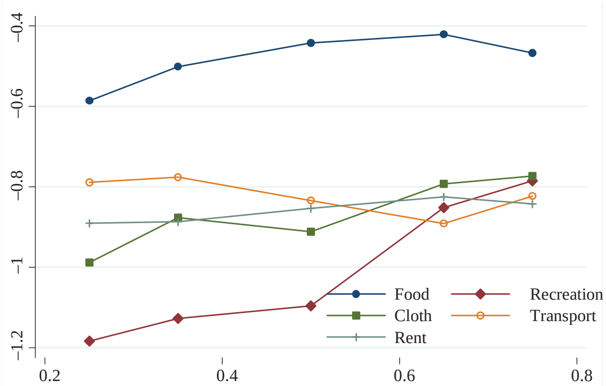

Figure 1. Generalized ML estimates as computed at the 25th, 35th, 50th and 75th expectiles in the H_{tr} sample

Back to equation (7), looking at the case $i = j$, it is worth noting that the estimated values of β_i are quite small across the expectiles, as can be seen in Tables 2 to 4. Thus their impact on elasticity is negligible and $\eta_{ii}(\theta)$ becomes a function of the ratio $\frac{\gamma_{ii}(\theta)}{\bar{w}_i}$, since $\tau_{ii} = 1$. Therefore, the larger this ratio the greater is the discrepancy of $\eta_{ii}(\theta)$ from unity.

Figure 1 complements Table 5 by depicting the pattern of each elasticity across the expectiles.[14] Figure 2, by reversing the axis, shows the changes in elasticity of the goods across expectiles, with Recreation presenting the widest variation, followed in decreasing order by Clothing, Food, and then Transport and Rent. These variations are reported in Table 6, where the differences between the estimated elasticity at the 25th, 50th and 75th expectile are computed. This table clearly shows the sign and the size of changes in Food, Recreation and Clothing elasticity across the expectiles, together with the comparatively small changes in Rent and Transport elasticity, thus providing evidence of their greater stability.[15]

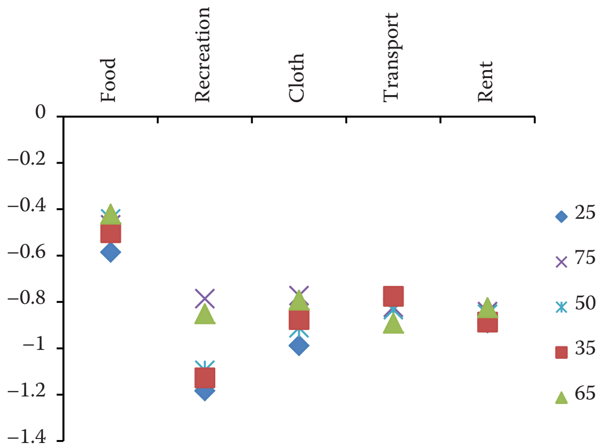

Figure 2. Estimates at each expectile, truncated sample

Table 6. Inter-expectiles differences in estimated elasticity

Sample H_{tr}	50th–25th	75th–50th	75th–25th
Food-in	0.14	–0.03	0.12
Recreation	0.09	0.31	0.40
Clothing	0.08	0.14	0.22
Transport	–0.04	0.01	–0.03
Rent	0.04	0.01	0.05

Figure 3 depicts the distribution of the elasticity of the five goods as estimated by the bootstrap at the 25th, 50th and 75th expectile. These bootstrap estimates are needed to compute the covariance matrix across expectiles Σ in the Wald test function of equation (11). However, their empirical distributions are quite informative and validate the above results. Food elasticity at the 25th expectile differs from the distributions at the 50th and 75th expectile. Clothing and Recreation elasticity at the 25th and at the 50th expectile are close to each other, but these curves are far from the distribution at the 75th expectile, signalling

Table 7. Wald test

Sample H_{tr}	50th–25th	75th–50th	75th–25th
Food-in	6.98***	0.07	0.98
Recreation	1.05	2.68*	3.08*
Clothing	0.78	0.64	0.95
Transport	0.40	0.01	0.05
Rent	1.76	0.08	0.76

The stars signal the significance level for the rejection of the null; H_0: $\boldsymbol{\eta}(\theta_v) = \boldsymbol{\eta}(\theta_m)$; ***for $\alpha = 0.01$,**for $\alpha = 0.05$, *for $\alpha = 0.10$

[14]Table 5 can be compared with Table A.2 of the Appendix, which reports the estimated elasticity in the original sample of size H. The original sample confirms the presence of changing parameters, although the estimates are somewhat different, smaller for food and larger for clothing and recreation. The comparison of Table 5 with Table A.2 and of Figure 1 with Figure A.1 shows that the presence of outliers causes greater heterogeneity in the behavior of each equation across quantiles in the original sample, as for instance occurs for Food and Clothing. This heterogeneity can hardly be explained by economic reasons and can be ascribed to the lack of robustness of the expectile estimator in the presence of outliers.

[15]Table 6 and 7 can be compared with Table A.3 and A.4 in the Appendix.

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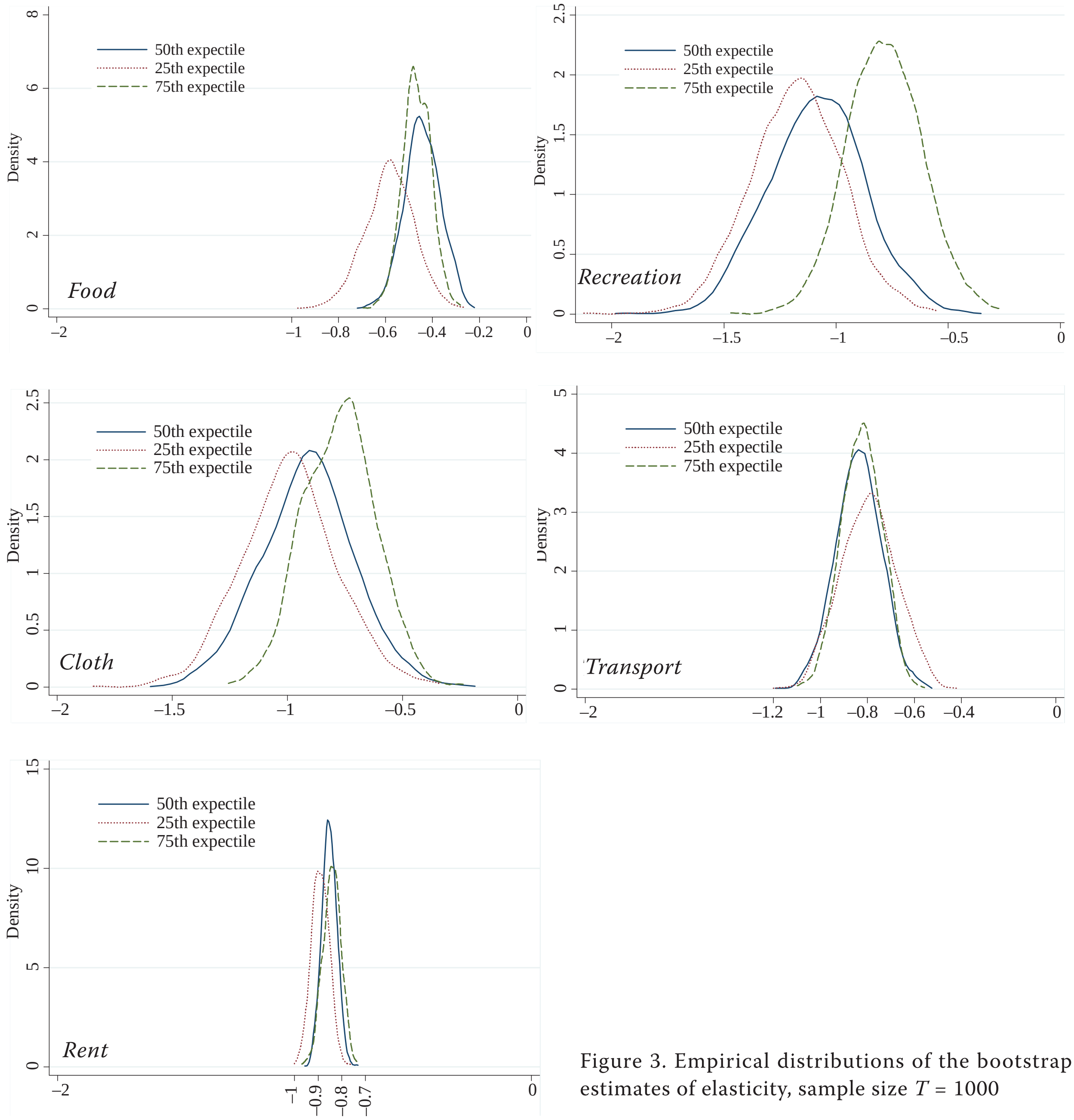

Figure 3. Empirical distributions of the bootstrap estimates of elasticity, sample size T = 1000

once again that these elasticities do change across the expectiles. Finally, Transport and Rent distributions do not change much across the expectiles.

Summarizing, the distributions of the bootstrapped elasticity provide an additional evidence of the presence of changing elasticity in three out of five goods and purport the relevance of an analysis implemented not only at the average but also in the tails of the conditional distribution.

Table 6 reports the difference of elasticity as computed for the five goods at two different expectiles, $\boldsymbol{\eta}(\theta_v) - \boldsymbol{\eta}(\theta_m)$. The table shows that the largest changes occur in Food – when comparing the elasticity at the 25th versus the 50th expectile; Recreation – in the comparison of the 25th versus the 75th expectile and of the 50th versus the 75th expectile; Cloth – in the comparison of the 25th versus the 75th expectile.

Finally, Table 7 reports the estimated Wald test for each equation comparing the 50th versus the 25th, the 75th versus the 50th, and the 75th versus the 25th expectile under the null H_0: $\boldsymbol{\eta}(\theta_v) = \boldsymbol{\eta}(\theta_m)$. These values are compared with the critical values of a χ^2 with one degree of freedom. The starred values signal the rejection of the null of constant elasticity

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at the significance level α = 1%, 5% and 10%. Food elasticity computed at the 25[th] and 50[th] expectile yields the largest estimated Wald test function, and the null is strongly rejected at α = 1%. Comparison of Recreation elasticity at the 50[th] versus the 75[th], and at the 25[th] versus the 75[th], allows the null at α = 10% to be rejected. Clothing elasticity, which in Figure 3 and in Table 6 does present discrepancies between the 25[th] versus the 75[th] and between the 50[th] and the 75[th] expectile, yields the Wald test results that do not reject the null. This may be due to the relatively larger dispersion of these distributions that offsets the changes in location at the various expectiles.[16]

CONCLUSIONS

A linear demand equation system is computed at different points of the conditional distribution through the expectile regression estimator. The latter provides a tool to investigate the existence of changing PED at different levels of expenditure. The selected model comprises five goods: Food, Recreation, Clothing, Transport and Rent, using the Canadian Family Expenditure Survey data in Lewbel and Pendakur (2009). The purpose of the analysis is not to re-estimate the PED values, but to highlight the opportunity to provide the PED estimates at the various levels of expenditures. Through the expectile estimator, the model can be computed not only at the mean, as is generally done when implementing the OLS, but also at the lower and higher levels of expenditure. By estimating the model only at the conditional average of expenditure, the behaviour of the demand equations in the tails is unknown and undefined.

The results show that the elasticity does change along the conditional distribution for three of the five goods under study. A Wald test on the equality of elasticity across the expectiles shows that these changes are statistically relevant for Food and Recreation. By contrast, Transport and Rent are more stable and none of the different approaches implemented provides evidence of instability in their elasticity along the conditional distribution. The above analysis can be extended in more than one direction, and a further research may focus on a larger number of goods, on the definition of nonlinear demand equations, on different robust regression estimators for the outlier detection.

APPENDIX

Table A.1. Estimates of the demand equations, original sample

	Food-in	Recreation	Clothing	Transport	Rent
α_i	0.422	–0.139	–0.093	0.239	0.571
	32.24	*–10.73*	*–8.47*	*20.12*	*30.98*
β_i	–0.066	0.067	0.050	–0.007	–0.043
	–20.26	*20.72*	*18.41*	*–2.32*	*–9.27*
$\gamma_{i,j: \text{Food-in}}$	0.095	–0.041	–0.062	–0.024	0.032
	7.36	*–3.8*	*–5.77*	*–2.52*	*2.97*
$\gamma_{i,j: \text{Recreation}}$	–0.041	0.012	0.079	–0.023	–0.027
	–3.8	0.68	5.46	–2.11	–2.26
$\gamma_{i,j: \text{Clothing}}$	–0.062	0.079	0.031	0.019	–0.068
	–5.77	*5.46*	*1.8*	*1.89*	*–6.47*
$\gamma_{i,j: \text{Transport}}$	–0.024	–0.023	0.019	0.032	–0.004
	–2.52	*–2.11*	*1.89*	*2.27*	*–0.34*
$\gamma_{i,j: \text{Rent}}$	0.032	–0.027	–0.068	–0.004	0.066
	2.97	*–2.26*	*–6.47*	*–0.34*	*3.67*
$\delta_{i,k: \text{Age}}$	0.001	–0.012	0.028	0.019	0.002
	13.82	*–5.17*	*10.36*	*6*	*6.06*
$\delta_{i,k: \text{Sex}}$	–0.001	–0.033	0.017	–0.004	–0.002
	–10.91	*–14.02*	*6.34*	*–1.39*	*–6.49*
$\delta_{i,k: \text{Car}}$		0.034	0.017	–0.017	–0.004
	–7.81	*17.7*	*7.64*	*–6.74*	*–18.5*
$\delta_{i,k: \text{Soc. ass.}}$	0.000	–0.014	–0.129	–0.015	–0.001
	–4.34	*–6.38*	*–53.56*	*–5.24*	*–1.76*
$\delta_{i,k: \text{Time}}$	0.001	0.024	0.067	0.017	0.005
	4.86	*7.08*	*17.26*	*3.81*	*14.23*

t-statistics in italics

Table A.2. Estimated elasticity at the selected expectiles, original sample *H*

Sample *H*	25[th]	35[th]	50 [th]	65 [th]	75 [th]
Food-in	–0.419	–0.487	–0.385	–0.479	–0.566
	(0.107)	*(0.106)*	*(0.083)*	*(0.058)*	*(0.065)*
Recreation	–1.039	–0.942	–0.903	–0.782	–0.834
	(0.211)	*(0.233)*	*(0.231)*	*(0.189)*	*(0.199)*
Clothing	–0.81	–0.992	–0.775	–0.464	–0.668
	(0.209)	(0.224)	(0.205)	(0.155)	(0.164)
Transport	–0.831	–0.793	–0.79	–0.878	–0.82
	(0.118)	(0.123)	(0.101)	(0.08)	(0.092)
Rent	–0.869	–0.884	–0.848	–0.87	–0.88
	(0.04)	(0.046)	(0.033)	(0.034)	(0.043)

Bootstrapped standard deviation in parenthesis

[16]This table can be compared with Table A.4 in the Appendix, which presents comparable results.

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Table A.3. Inter-expectiles differences in the estimated elasticity, original sample *H*

Sample H	50th–25th	75th–50th	75th–25th
Food-in	0.03	–0.18	–0.15
Recreation	0.14	0.07	0.20
Cloth	0.04	0.11	0.14
Transport	0.04	–0.03	0.01
Rent	0.02	–0.03	–0.01

Table A.4. Wald test, original sample *H*

Sample *H*	50th–25th	75th–50th	75th–25th
Food-in	8.45***	0.27	0.93
Recreation	1.73	3.52*	4.33*
Cloth	1.00	1.44	1.84
Transport	0.48	0.21	0.00
Rent	2.01	0.44	1.45

The stars signal the significance level of the rejection of the null H_0: $\eta(\theta_v) = \eta(\theta_m)$; ***for $\alpha = 0.01$, **for $\alpha = 0.05$, *for $\alpha = 0.10$

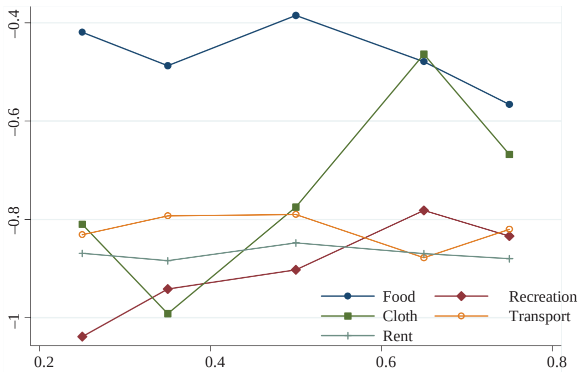

Figure A.1. Estimated elasticity at the selected expectiles, original sample *H*. The presence of outliers causes greater heterogeneity in the behaviour of each equation across quantiles

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